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**Comments on Shimomura-Odajima Scaling**

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## Comments on Shimomura–Odajima Scaling

In Section I the empirical Shimomura–Odajima “linear offset” scaling for the global energy confinement  $\tau^G$  is analyzed from a theoretical point of view. Eliminating the temperature dependence of the ohmic power, positive scaling of  $\tau^G$  with plasma current and a parameter  $\hat{\tau}_d^{-1} = 1.6ARn_{20}$  emerges, where  $A$  is the aspect ratio,  $R$  the major radius in meters, and  $n_{20}$  the plasma density in units of  $10^{20}/\text{m}^{-3}$ . In the moderately large auxiliary power limit,  $\tau^G = \tau_d + \tau_\Omega k I_p^{0.8}/P_{\text{aux}}$ . Here  $\tau_d = \hat{\tau}_d \tau_\Omega$  and  $\tau_\Omega$  is the ohmic confinement time. An explicit reformulation is given for  $\tau^G$  as a function of auxiliary (not total) power  $P_{\text{aux}}$ , i.e.,  $\tau/\tau_\Omega = (1 + \hat{\tau}_d P)/(P + (1 + \hat{\tau}_d P)^{-3/2})$ ,  $P = P_{\text{aux}}/P_\Omega^0$  and  $P_\Omega^0$  is the ohmic power before  $P_{\text{aux}}$  is applied. This scaling favors high magnetic field tokamaks with large  $ARn$ . In Section II, we use the method of Callen, Cordey *et al.* (deriving the linear offset scaling law formally from the averaged power balance) to exhibit the justification and limitations of the Shimomura–Odajima ansatz for  $\tau^G$ .

**Key Words:** tokamak global scaling laws, tokamak energy confinement, tokamak particle confinement, auxiliary heating, fueling

### I. EMPIRICAL LINEAR OFFSET SCALING

#### Introduction

The search for a theoretical understanding of the empirical scaling of the energy confinement time is of central importance for design of an ignition experiment. While present experimental data with present auxiliary power levels can be fitted equally well with Goldston's  $\tau_E \sim P^{-1/2}$  or Doublet-JET scaling  $\tau_E \sim a + b/P$ , the extrapolation to the large heating and alpha power levels of ignition

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devices requires a decision on the correct functional dependence. Recently, Shimomura–Odajima *et al.*<sup>1,2</sup> (henceforth S-O) have prepared a parameterization of a large experimental data base in the form

$$\tau^G = \tau_\Omega \frac{P_\Omega^0}{P_t} + \tau_d \frac{P_t - P_\Omega^0}{P_t} = \frac{P_\Omega^0}{P_t} (\tau_\Omega - \tau_d) + \tau_d. \quad (1)$$

Here, one may take

$$\tau_\Omega = 0.07 \bar{n}_{20} a R^2 \sqrt{\kappa} q, \quad (2a)$$

the neo-Alcator ohmic confinement time in the unsaturated regime, or the optimized “scaling”

$$\tau_\Omega^{\text{op}} = 0.045 R a B_i \kappa^{0.5} A_i^{0.5} \quad (2b)$$

and

$$\tau_d = 0.12 a^2 \quad (3)$$

which is the incremental or differential confinement time of S-O. (The dimensions are seconds, density in units of  $10^{20} \text{ m}^{-3}$ , and meters.  $A_i$  is the atomic mass.)  $P_\Omega^0$  is the ohmic power *before* auxiliary heating is applied<sup>3</sup> at an initial ohmic temperature  $T_o$  and

$$P_t = P_\Omega^0 \left( \frac{T}{T_o} \right)^{-3/2} + P_{\text{aux}} \quad (4)$$

is the total power at the prevailing temperature  $T$  after  $P_{\text{aux}} > 0$  has been applied.

Equation (3) is the main result of Ref. 1 and is given there as a convenient simplification of their regression formula

$$\tau_d = (0.1) R^{0.06} a^{1.79} \kappa^{0.42} B_i^{0.21} q^{-0.22} \approx \left[ (0.1) R^{0.06} \kappa^{0.42} \left( \frac{B}{aq} \right)^{0.2} \right] a^2.$$

## Discussion

A heuristic justification of Eq. (1) is as follows. Since the stored energy  $W$  should be a monotonically increasing function of  $P_{\text{aux}}$  (as long as the plasma remains stable) one may define an incremental confinement time

$$\tau_{\text{inc}} \equiv \frac{\Delta W}{\Delta P_{\text{aux}}} = \frac{W - W_\Omega}{P_t - P_\Omega} = \frac{W - \tau_\Omega P_\Omega}{\Delta P_{\text{aux}}}. \quad (5)$$

Here we used

$$\tau_\Omega \equiv \frac{W_\Omega}{P_\Omega} \quad (6)$$

and

$$W = W_\Omega + \Delta W = \tau_\Omega P_\Omega + \tau_{\text{inc}} \Delta P_{\text{aux}}. \quad (7)$$

Defining the global energy confinement time

$$\tau^G \equiv \frac{W}{P_t}$$

the form of Eq. (1) is recovered.  $\tau_\Omega$  is taken from ohmically heated discharges, cf. Eqs. (2a,b). Thus the key question is the scaling of  $\tau_{\text{inc}}$  with  $P_{\text{aux}}$ , the plasma current  $I_p$  and other parameters. In Section II, we will provide theoretical constraints on  $\tau_d$ . Here, we first address the apparent disadvantage of Eq. (1) where  $P_\Omega$  depends on the self-consistent value of the plasma temperature  $T$  (and its scaling on plasma parameters).

## Hidden Scaling with Plasma Current and Density

All other existing scaling laws for  $\tau^G$  with  $P_{\text{aux}}$  show a positive scaling with plasma current  $\tau^G \propto I_p^\nu$  where  $\nu \gtrsim 1$  [cf. Refs. 4–6]. This can be reconciled with Eq. (1) (where  $P_t$  is used) as follows. Since from Eq. (4)  $P_t$  contains the plasma temperature, Eq. (1) is

not complete without simultaneously solving for the temperature from

$$\frac{nTv}{\tau^G} = P_\Omega^\circ \left( \frac{T}{T_o} \right)^{-3/2} + P_{aux}. \quad (8)$$

Here,  $v$  is the plasma volume. We define the dimensionless quantities

$$\hat{\beta} = nT/nT_o \equiv \beta/\beta_\Omega \quad (9a)$$

$$P = P_{aux}/P_\Omega^\circ \quad (9b)$$

$$\tau_\Omega = nT_o v / P_\Omega^\circ \quad (9c)$$

$$\hat{\tau} = \tau^G / \tau_\Omega \quad (9d)$$

$$\hat{\tau}_d = \tau_d / \tau_\Omega^{\text{op}} = (0.63)(a/R)(n_{20}R)^{-1} \quad (9e)$$

or

$$\hat{\tau}_d = \tau_d / \tau_\Omega^{\text{op}} = 2.7 \left( \frac{a}{R} \right) (B_i \kappa^{0.5} A_i^{0.5})^{-1} \quad (9f)$$

and obtain from Eqs. (8), (9) and (1) their dimensionless forms

$$\frac{\hat{\beta}}{\hat{\tau}} = P + \hat{\beta}^{-3/2} \quad (10a)$$

$$\hat{\tau} = \frac{1 - \hat{\tau}_d}{\hat{\beta}^{-3/2} + P} + \hat{\tau}_d. \quad (10b)$$

These equations determine the normalized confinement time  $\hat{\tau}$  and stored energy  $\hat{\beta}$  as a function of auxiliary (not total) normalized power  $P$ .  $\hat{\tau}_d$  is a free parameter (see Eqs. (9e,f)).

An intermediate result for  $\hat{\beta} = \hat{\beta}(\hat{\tau})$  emerges from (10a,b):

$$\hat{\beta} = (1 - \hat{\tau}_d) \left( 1 - \frac{\hat{\tau}_d}{\hat{\tau}} \right). \quad (11)$$

Equations (10) have the decoupled inversions

$$P = \frac{\hat{\beta}}{\hat{\tau}_d} - \hat{\beta}^{-3/2} - \frac{1}{\hat{\tau}_d} + 1 \quad (12a)$$

and

$$P = \left( \frac{1}{\hat{\tau}_d} - 1 \right) \left( \frac{1}{y} - 1 \right) - (1 - \hat{\tau}_d)^{-3/2} y^{3/2} \quad (12b)$$

where  $y \equiv 1 - \hat{\tau}_d/\hat{\tau}$  is related to the confinement time.

Since  $\hat{\tau}_d$  (defined in Eqs. (9)) is smaller than one for all tokamaks of interest (e.g.,  $\hat{\tau}_d = 0.47$  for ISX-B,  $\hat{\tau}_d = 0.02$  to  $0.05$  for CIT) and  $\hat{\beta} \equiv \beta/\beta_\Omega \geq 1$  for  $P_{aux} \geq 0$ , Eq. (12a) has the dominant solution

$$\hat{\beta} \approx 1 + \hat{\tau}_d [P + (1 + \hat{\tau}_d P)^{-3/2} - 1] \quad (13a)$$

whence from Eq. (10b)

$$\hat{\tau} \approx \frac{1 - \hat{\tau}_d}{P + (1 + \hat{\tau}_d P)^{-3/2}} + \hat{\tau}_d. \quad (13b)$$

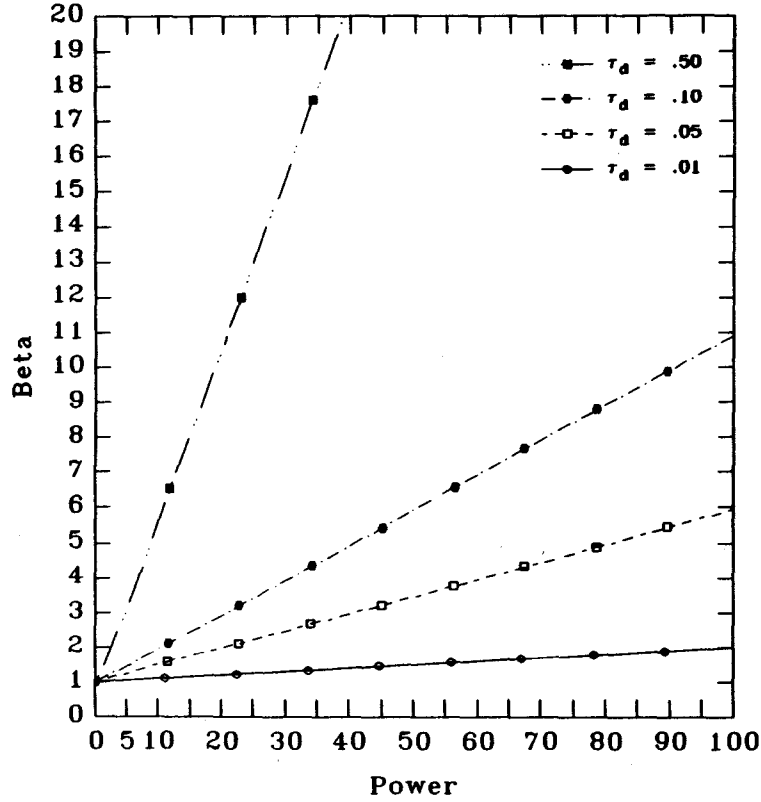
Our main results of Eqs. (13) are useful analytic approximations to the exact numerical solutions for  $\hat{\beta}(P; \hat{\tau}_d)$  and  $\hat{\tau}(P; \hat{\tau}_d)$  defined by Eqs. (10) and given in Figs. 1 and 2. [Note that in the figures the dimensionless confinement time plotted is  $t \equiv \hat{\tau}/\hat{\tau}_d$ , not  $\hat{\tau}$ , and the horizontal axis shows the normalized auxiliary power of Eq. (9b).] Equation (13b) gives the confinement time explicitly as a function of  $P \equiv P_{aux}/P_\Omega^\circ$ , rather than  $P_t = P_\Omega(T) + P_{aux}$ .

In the ohmic heating limit  $\hat{\tau}_d P \rightarrow 0$ , Eq. (13b) becomes

$$\hat{\tau} \rightarrow \frac{1}{1 + P} + 0(\hat{\tau}_d P)$$

or

$$\tau \rightarrow \tau_\Omega \left( 1 + \frac{P_{aux}}{P_\Omega^\circ} \right)^{-1}. \quad (14a)$$



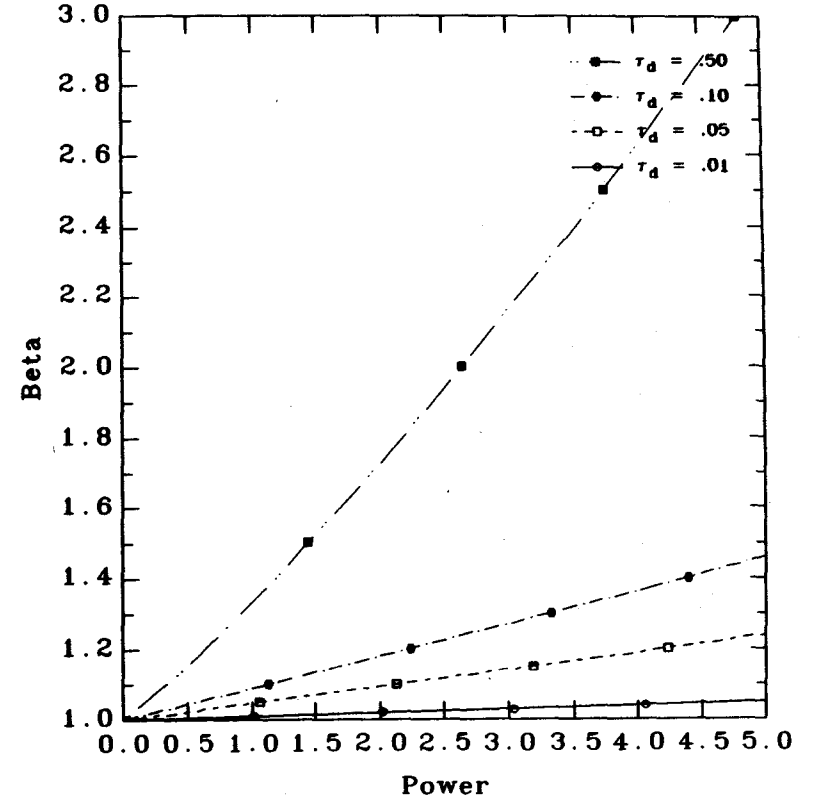
(a)

In the strong heating limit,  $\hat{\tau}_d P > 1$ , one obtains

$$\hat{\tau} \rightarrow \hat{\tau}_d + \frac{1}{P} (1 - \hat{\tau}_d) (1 - \hat{\tau}_d^{-3/2} P^{-5/2} \dots)$$

or, dropping the last term and assuming  $\hat{\tau}_d \ll 1$

$$\tau \rightarrow \tau_d \left[ 1 + \frac{P_\Omega^0}{P_{\text{aux}}} \frac{\tau_\Omega}{\tau_d} \right]. \quad (14b)$$



(b)

FIGURE 1 (a) Stored energy versus auxiliary heating power. Specifically,  $\hat{\beta} \equiv \beta/\beta_\Omega = T/T_0$  versus  $P = P_{\text{aux}}/P_\Omega^0$ , with values of  $\hat{\tau}_d$  (defined in Eq. (9e,f)) varying from 0.01 to 0.05. Equation (12a) was used. (b) More detail for  $0 \leq P \leq 5$ .

In the intermediate power regime  $\hat{\tau}_d P \sim 0(1)$ , the full Eq. (13b) has to be used. We note from (13a) that the stored energy  $\beta$  is approximately linear with the variable  $P = P_{\text{aux}}/P_\Omega^0$  (see Fig. 1). The dependence on plasma current is contained in Eqs. (13) and (14) through  $P_\Omega^0 = \int d^3x \mathbf{j} \cdot \mathbf{E} = k I_p^{0.8}$ , where the constant  $k$  depends on plasma parameters. Thus, from (14b),

$$\tau \sim \tau_d + \tau_\Omega k I_p^{0.8} / P_{\text{aux}}. \quad (14c)$$

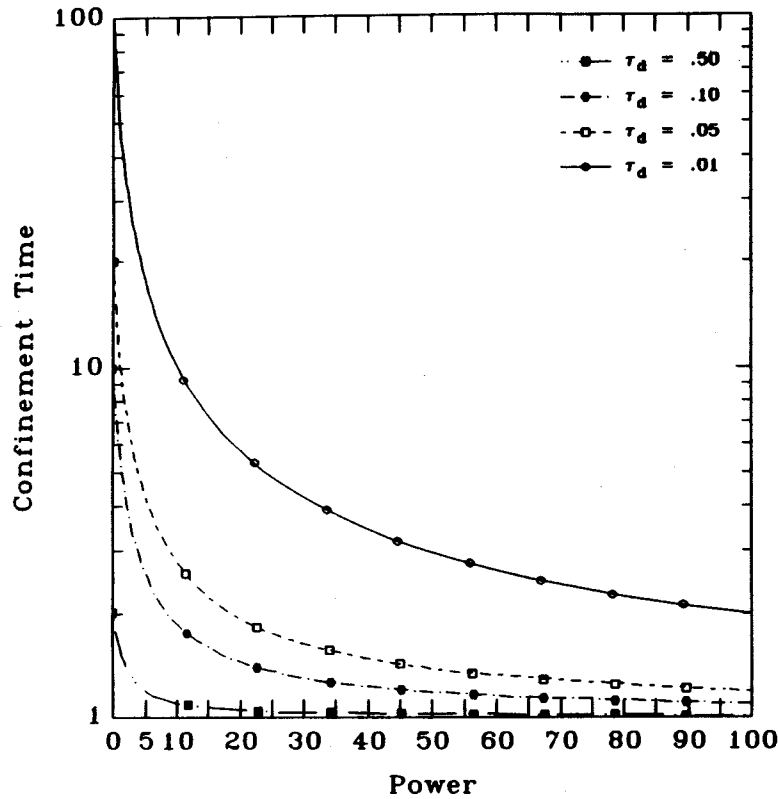


FIGURE 2 Normalized energy confinement time versus normalized power. Specifically,  $\hat{\tau}/\hat{\tau}_d$  versus  $P = P_{aux}/P_{\Omega}^0$ , for  $\hat{\tau}_d$  values between 0.01 and 0.5. Equation (12b) was used. Logarithmic scale for  $\hat{\tau}/\hat{\tau}_d$ , linear scale for  $0 \leq P \leq 100$ .

The exact current dependence at fixed  $P_{aux}$  is shown in Fig. 5 for CIT parameters (details in Appendix A).

In all cases, the current dependence enters through  $P = P_{aux}/P_{\Omega}^0$ . As can be seen from Fig. 2, increasing the plasma current  $I_p$  at fixed  $P_{aux}$  increases the energy confinement time  $\hat{\tau}$  by moving to smaller values of  $P = P_{aux}/P_{\Omega}^0$ , since  $P_{\Omega}^0 \propto I_p^{0.8}$ . For example, for CIT parameters keeping  $P_{aux}$  fixed at 10 MW,  $\hat{\tau}_d = 0.02$  and

varying  $I_p$  from 3 MA to 11 MA increases  $\hat{\tau}$  by a factor of 2 (see Fig. 5).

If, in addition, the parameter  $\hat{\tau}_d \sim (AnR)^{-1}$  is decreased one jumps to a different curve in Fig. 2 with generally larger  $\hat{\tau}$ . For example, Figs. 2 and 3 show the favorable dependence of  $\hat{\tau}$  on  $\hat{\tau}_d$ , for fixed  $P = 20$ , going from ISX-B ( $\hat{\tau}_d = 0.47$ ) to CIT ( $\hat{\tau}_d = 0.2$ ). Figure 4 shows the strong improvement of confinement time with  $\hat{\tau}_d^{-1}$ . Using the Murakami density limit  $n = 0.7B/R$ ,  $\hat{\tau}_d^{-1} = 1.12BA$ . The high current possible at high field provides

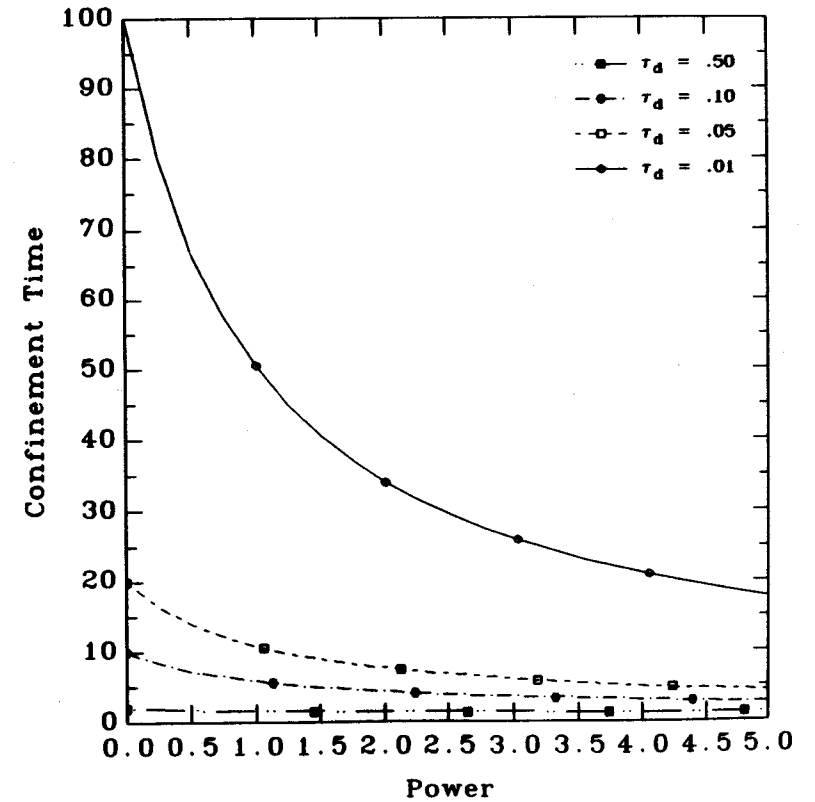


FIGURE 3 Normalized energy confinement time versus power, same as in Fig. 2, but the linear scale for  $\hat{\tau}/\hat{\tau}_d$  shows the expanded range  $0 \leq P \leq 5$ .

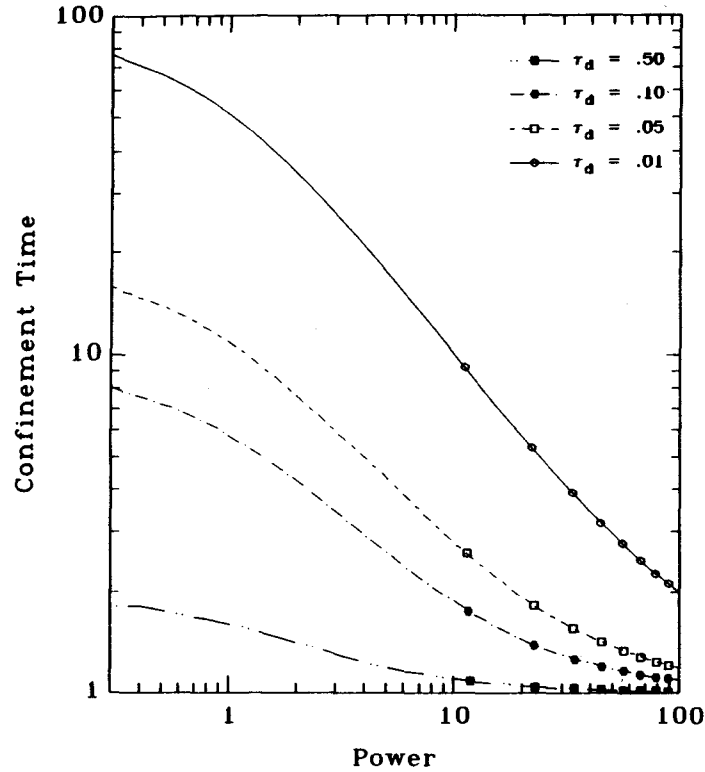


FIGURE 4 Same as Fig. 2, in a log-log plot. Note the strong improvement of energy confinement time with  $\tau_d^{-1}$ , thus favoring high field/high density tokamaks.

high ohmic power, thereby reducing the variable  $P$  at fixed  $P_{aux}$ . These results, which are contained analytically in Eq. (13b), clearly favor the high field and thus the high ohmic power high density approach to tokamak confinement.

## II. THEORETICAL JUSTIFICATION FOR OFFSET SCALING LAW

Recently, Callen, Cordey *et al.*<sup>7</sup> have given a theoretical derivation of the global energy confinement time from the volume averaged

energy balance equation which under certain circumstances leads to a linear offset scaling law. Here, we generalize this method to multiple ion species and display the conditions for which a Shimomura-Odajima type of scaling law is recovered.

Defining the stored energy

$$\bar{W} \equiv L \int_0^a dr r \sum n_j T, \quad j = e, i, z \quad (15)$$

where  $L = 4\pi^2 R_0$  and we assume the individual species temperatures to have equilibrated and  $T(r = a) = 0$ , integration by parts yields  $\bar{W}/L = -\int_0^a dr z(r) \partial T / \partial r$ . Here,

$$z(r) \equiv \int_0^r dr' r' \sum n_j(r') \quad (16a)$$

is assumed to be known. (A fuller treatment involving the particle balance is given in Appendix C). Introducing the thermal conductivity

$$\kappa(r) \equiv \sum_j n_j \chi_j \quad (16b)$$

where  $\chi_j$  is the thermal diffusivity of species  $j$ , we define the variable

$$y(r) \equiv \int_0^r \frac{dr' z(r')}{r' \kappa(r')}, \quad y(a) \equiv \tau_\kappa \quad (16c)$$

and obtain

$$\frac{\bar{W}}{L} = - \int_0^a dr \frac{dy}{dr} \left( r \kappa \frac{\partial T}{\partial r} \right). \quad (17)$$

Note that  $\tau_\kappa$  scales as  $a^2/(4\bar{\kappa}/\bar{n})$ , a profile averaged confinement time. Also note that  $dy/dr$  is a positive definite function of  $r$ . The flux surface averaged energy balance equation summed over all species has the general conservation form

$$\frac{1}{r} \frac{\partial}{\partial r} r \left[ -\kappa \frac{\partial T}{\partial r} + q_c \right] = Q, \quad (18)$$

where  $q_c$  is the total convective radial heat flux and  $Q$  the net total power source per unit volume, including ohmic and all auxiliary heating, radiation and charge exchange losses. ( $Q$  could also contain  $-(\partial/\partial t)/(3/2) \sum n_i T$ , if necessary.) Inserting (18) in (17) yields

$$\frac{\bar{W}}{L} = \tau_\kappa \int_0^a dr r Q \left( 1 - \frac{y(r)}{y(a)} \right) - \int_0^a dr r \frac{dy}{dr} q_c. \quad (19)$$

The global energy confinement time  $\tau_E \equiv \bar{W}/P_t$  becomes

$$\tau_E = \tau_\kappa \left\{ \frac{\int_0^a dr r Q (1 - y(r)/y(a))}{\int_0^a dr r Q} - \frac{L \int_0^a dr r [1/y(a)] (dy/dr) q_c}{P_t} \right\}, \quad (20a)$$

where the total power

$$P_t \equiv L \int_0^a dr r Q, \quad \text{with } Q = Q_\Omega + Q_{\text{aux}} + \dots \quad (20b)$$

was introduced. Thus,  $P_t$  includes the ohmic power as in Section I.

#### Discussion

Equation (19) is of the anticipated offset form

$$\bar{W} = \tau P_{\text{aux}} + \bar{W}_0, \quad (20c)$$

where  $\bar{W}_0 > 0$  (see Appendix B). Thus, the first term in Eq. (19) is proportional to the heating power weighted by a form factor depending on the ratio of the sum over species heat conductivity. The second term (which could be positive if there is a strong "heat pinch") depends entirely on the existence of a convective piece of the total radial heat flow, summed over species.

The "convective heat flux"  $q_c$  is defined here as the sum of all contributions to the total heat flux minus the diagonal diffusive term  $-n\chi \partial T/\partial r$ . Thus,  $q_c$  contains all off-diagonal terms of the Onsager matrix plus the pure heat convection  $(5/2)\Gamma T$ , where  $\Gamma$  is

the radial particle flux (cf. Ref. 8(a), Eqs. (167) to (173).) Since all realistic tokamak plasmas have  $Z_{\text{eff}} > 1$  such that  $n_z Z^2/n_i \gg \sqrt{m_e/m_i}$ , it follows that the neoclassical main ion particle flux is determined by ion-impurity rather than ion-electron friction and there results (cf. Ref. 8(b)), in the absence of ion turbulence

$$\Gamma_i = -D_{iz} n_i \left[ \frac{n'_i}{n_i} - \frac{1}{Z} \frac{n'_z}{n_z} + k(v_*, \epsilon) \frac{T'}{T} \right]$$

$$\Gamma_e = \Gamma_e^{\text{anom}} - 2.44 \sqrt{\epsilon n} \frac{E_\parallel}{B_\theta}$$

$$\Gamma_z = \frac{1}{Z} (\Gamma_e - \Gamma_i),$$

the ambipolar condition. Here,  $D_{iz} \sim \sqrt{\epsilon} \rho_{pi}^2 v_{iz} \sim \chi_i$  (neoclassical), ' denotes  $\partial/\partial r$  and  $k(v_*, \epsilon)$  is a coefficient of  $O(1)$ , which can change sign [see Ref. 8(b)]. With  $D_{iz} \sim \chi_i$  (neoclassical) the convective term  $(5/2)\Gamma_i T$  can compete with the diagonal diffuse heat flux term, thus guaranteeing a nonvanishing contribution from  $q_c$  in Eq. (20a). Indeed, using ambipolarity to eliminate  $\Gamma_z$

$$q_c = \sum_i \frac{5}{2} \Gamma_i T \approx \frac{5}{2} T \left[ \left( 1 + \frac{1}{Z} \right) \Gamma_e^{\text{anom}} + \left( 1 - \frac{1}{Z} \right) \Gamma_i \right] - 6.1 \sqrt{\epsilon n} T \frac{E_\parallel}{B_\theta} \left( 1 + \frac{1}{Z} \right) + \text{off-diagonal terms}. \quad (21)$$

The last (inward directed) term increases the stored energy, but is typically not big enough to reverse the sign of the whole expression. Zarnstorff *et al.*<sup>10</sup> have measured  $q_c$  in TFTR. In addition, there are the (neoclassical and anomalous)<sup>11</sup> off-diagonal contributions to  $q_c$ , including an outward directed Ware heat flux,  $1.75 \sqrt{\epsilon n} T (E_\parallel/B_\theta)$ , cf. Ref. 8(a), which opposes the Ware term in (21).

Three important facts emerge:

- (i) The degradation of  $\tau_E$  is not necessarily predicated on an anomalous increase of  $\chi_i$  and  $q_{ci}$  due to increased heating power.



Rather, the level of stored energy  $\bar{W}$  of Eq. (19) achieved with application of a given  $P_r \sim \int dr r Q$  can be likened to (a) filling a water bucket with diffuse walls ( $\tau_\kappa < \infty$ , first term on right-hand side of (19)) and (b) having a hole/convective loss (second term on the right-hand side of (19)). Since, fundamentally,  $q_c \propto nT v_r$ , the convective loss increases proportionally to the stored energy already achieved. This causes the dependence  $\tau_E \sim b/P$ .

- (ii) Each individual term on the right-hand side of Eq. (20a) is or may be implicitly temperature dependent, e.g.,  $P_\Omega$ . This necessitates a return to the procedure used on the Shimomura–Odajima scaling law in Section I, namely to eliminate the temperature dependence by reusing the energy balance Eq. (18). This yields

$$T = T(P_{\text{aux}}; P_n) \quad (22)$$

where  $P_n$  stands for all other plasma parameters. Then, inserting (22) into all temperature dependent terms of Eq. (20a) produces a subsidiary dependence on  $P_{\text{aux}}$  and reveals that (20a) will not in general be a *linear* offset scaling law.

- (iii) Equation (20) contains the exact transition from ohmic to strong auxiliary heating, albeit implicitly. This enables a rigorous test of the heuristic transition rule introduced by R. Goldston  $(1/\tau_E)^2 = (1/\tau_E^\Omega)^2 + (1/\tau_E^{\text{aux}})^2$ .

Comparison with the Shimomura–Odajima Ansatz for  $\bar{W}$

To exhibit the relationship between Eq. (1) and Eq. (20a) we take two limits. In the ohmic heating limit  $P_{\text{aux}} \rightarrow 0$

$$\frac{\bar{W}(\text{Eq. (20a)})}{L\tau_\kappa^\Omega} = \int_0^a dr r Q_\Omega \left[ 1 - \frac{y}{y_a} - \frac{1}{y_a} \frac{dy}{dr} \tau_\Omega v_r \right]^{(\Omega)} \quad (23a)$$

The superscript  $\Omega$  indicates that ohmic plasma transport quantities are used. Here, using  $nT/\tau_\Omega = Q_\Omega$ , we set  $q_c \sim nT v_r \sim Q_\Omega v_r$  and

$v_r$  is the sum over species radial particle flow velocity. With  $L \int_0^a dr r Q_\Omega = P_\Omega$ , and recalling  $\tau_\kappa = y_a$ , Eq. (23a) becomes

$$\bar{W}(\text{Eq. (20a)}) = P_\Omega \tau_\kappa^\Omega \left[ 1 - \frac{y}{y_a} - \frac{\tau_\Omega}{\tau_\kappa^\Omega} \frac{dy}{dr} v_r \right]_{\text{av}}^\Omega \quad (23b)$$

where the subscript *av* indicates a radial average.

Equation (23b) describes the microscopic content of the simple term  $\bar{W}_\Omega = P_\Omega \tau_\Omega$  of Shimomura–Odajima in this limit  $P_{\text{aux}} \ll P_\Omega$ . Note that (23b) contains diffusive *and* convective contributions.

In the large auxiliary power limit  $P_{\text{aux}}/P_\Omega \rightarrow \infty$ ,

$$\begin{aligned} \bar{W}(\text{Eq. (20a)}) &= \tau_\kappa^\infty L \int_0^a dr r Q_{\text{aux}} \left( 1 - \frac{y}{y_a} \right)_\infty \\ &= P_{\text{aux}} \tau_\kappa^\infty \left( 1 - \frac{y}{y_a} \right)_{\text{av}}^\infty \end{aligned} \quad (24)$$

where the index  $\infty$  indicates the possibility that  $\chi_j^\infty \neq \chi_j^\Omega$ , etc. This could occur if strong auxiliary power drives the plasma to an instability threshold (e.g., ballooning mode,  $\eta_i$ - or  $\eta_e$ -mode) but there exists experimental evidence to the contrary, cf.  $\chi_e$  remaining the same in T-10 with strong ECH, cf. Ref. 9.

Thus, Eq. (24) reveals the meaning of  $\tau_d$  in Shimomura–Odajima's expression  $\bar{W}_{\text{incr}} = P_{\text{aux}} \tau_d$ , where  $\tau_d$  was given in Eq. (3).

We note from  $\tau_\kappa^\infty$  in Eq. (24) and (16c) a natural scaling of  $\tau_d$  with  $a^2$ , as Shimomura–Odajima observed, cf. Eq. (3). But, besides  $a^2$ ,  $\tau_\kappa^\infty$  allows for an additional scaling dependence of the incremental confinement time.

## CONCLUSIONS AND SUMMARY

In this paper, we have shown that if one accepts the quasi-heuristic ansatz Eq. (1) by Shimomura–Odajima and eliminates the implicit temperature dependence there results a nonlinear (in  $P_{\text{aux}}$ ) offset scaling law which shows a roughly linear increase of  $\tau_E$  with plasma current, for large enough currents.

The success of previous fits of  $\tau_E$  data either with a linear offset law or an inverse fractional power law is restricted to a relatively small range of auxiliary power (cf. Fig. 4). Over a fuller range ( $1 \leq P_{\text{aux}}/P_\Omega \leq 100$ ) Eq. (13b) shows the nonlinear  $P_{\text{aux}}$ -dependence of the Shimomura–Odajima scaling.

The dependence of  $\tau_E$  on the parameter  $\hat{\tau}_d \equiv 0.12a^2/\tau_\Omega$  (where  $\tau_\Omega$  is given by Eqs. (2a, 2b) depending on  $n_e \geq n_e^{\text{crit}}$ , the neo-Alcator saturation density) favors large major radius, high density (i.e., high magnetic field) machines.

Generalizing,<sup>7</sup> we find that Shimomura–Odajima's scaling of  $\tau_d$  with  $a^2$  has theoretical justification (via the factor  $\tau_\kappa$  defined in (16c)).

Our approach results in the form  $\tau_E = \tau_\kappa(1 - c_1 - c_2/P_i)$  given in Eqs. (20) where the constants  $\tau_\kappa$ ,  $c_1$  and  $c_2$  appear in terms of volume averaged elements of the Onsager matrix of local transport coefficients for each plasma species.

After choosing the set of these coefficients (from detailed experiments or theoretical results) their temperature dependence should be eliminated in favor of a  $P_{\text{aux}}$ -dependence (re-using the energy balance equation). This introduces subsidiary dependences on  $P_{\text{aux}}$ , just as it did in our treatment of the Shimomura–Odajima scaling law in Section I.

Thus, overall, a methodology has been developed which by eliminating implicit temperature dependences and using a first principles expression for  $W$  (from Ref. 7) yields a uniquely determined global energy confinement time, if the local transport coefficients are known. This method may be used to obtain a theory-based energy confinement law for an ignited plasma. From the appearance of the total summed over species power source term  $Q$  in Eq. (20) it is clear that the alpha power will contribute equally with  $P_{\text{aux}}$  to the degradation. Additional effects may arise from  $\chi_\alpha$  and  $q_{\text{ca}}$  (work in progress).

## APPENDIX A ON OHMIC POWER SCALING

In Appendix II of Ref. 2, S-O give an empirical scaling expression for the ohmic stored energy, namely

$$W_{\text{oh}}(\text{S-O}) = 0.095 I^{0.8} a^{0.8} R^{1.4} n^{0.6} B_i^{0.2} \kappa^{0.4} Z_{\text{eff}}^{0.2} A_i^{0.2}, \quad (\text{A.1})$$

where  $n$  is the average density.  $P_\Omega = W_{\text{oh}}/\tau_\Omega$  follows using Eq. (2a) or (2b) for  $\tau_\Omega$ . To check this analytically, we use

$$W_{\text{oh}} = P_\Omega \tau_{\text{oh}} \sim (\eta j^2 v)(n R^2 a q).$$

Ignoring current profile effects, we use  $j \sim (B/R_q)$ ,  $\eta \sim T^{-3/2}$ ,  $B = IRq/a^2$  and obtain

$$W_{\text{oh}} \sim I^{0.8} R^{1.8} a^{0.8} n/q^{0.4}. \quad (\text{A.2})$$

The factor  $n^{0.6}$  in Eq. (A.1) indicates that S-O included  $\tau_E$ -values in the saturated neo-Alcator regime, but aside from that, the scaling of  $W_{\text{oh}}$  in Eq. (A.1) with current and size is the same as in Eq. (A.2) and gives  $W_{\text{oh}} \sim I^{0.8}$ . For CIT parameters ( $I = 11$  MA,  $a = 0.7$  m,  $R = 2.1$  m,  $n_{20} = 5$ ,  $B_i = 11$  T,  $\kappa = 2$ ,  $Z_{\text{eff}} = 1.5$ ,  $A_i = 2$ ), we find

$$P_\Omega = k I^{0.8} \quad \text{where } k = 0.97 \quad (\text{A.3})$$

for the saturated formula (2b) for  $\tau_\Omega$ . Equation (A.3) is the formula used to construct Fig. 5, showing  $\tau_E = \tau_E(I)$  for  $P_{\text{aux}} = \text{const}$ . For these CIT parameters,  $W_{\text{oh}} = 9.6$  MJ,  $\tau_\Omega^{\text{op}} = 1.45$ ,  $P_\Omega = 6.6$  MW.

## APPENDIX B ON THE PROOF FOR $\bar{W} = \bar{W}_0 + \tau P_{\text{aux}}$ , $\bar{W}_0 > 0$

Fundamentally,  $\bar{W}_0 > 0$  is guaranteed from the energy balance (18), but it is worthwhile to study the detail in Eq. (19) which we scale as

$$\bar{W} = \tau_\kappa(P_\Omega + P_{\text{aux}}) \left(1 - \frac{y}{y_a}\right)_{\text{av}} - La\tau_\kappa n \bar{T} v_r. \quad (\text{B.1})$$

For  $P_{\text{aux}} \rightarrow 0$ ,  $\bar{W} \rightarrow \bar{W}_0$ , and using  $\bar{W} = (1/2)La^2 n \bar{T}$ , Eq. (B.1) becomes

$$\bar{W}_0(1 + 2\tau_\kappa^\Omega v_r/a) = \tau_\kappa^\Omega \left(1 - \frac{y}{y_a}\right)_{\text{av}}^{(\Omega)} P_\Omega.$$

With  $v_r/a = D/a^2 \equiv 1/\tau_p$

$$\bar{W}_0 = \frac{\tau_\kappa^\Omega (1 - y/y_a)_{\text{av}}^{(\Omega)} P_\Omega}{1 + 2\tau_\kappa^\Omega/\tau_p} \equiv \tau_\Omega P_\Omega. \quad (\text{B.2})$$

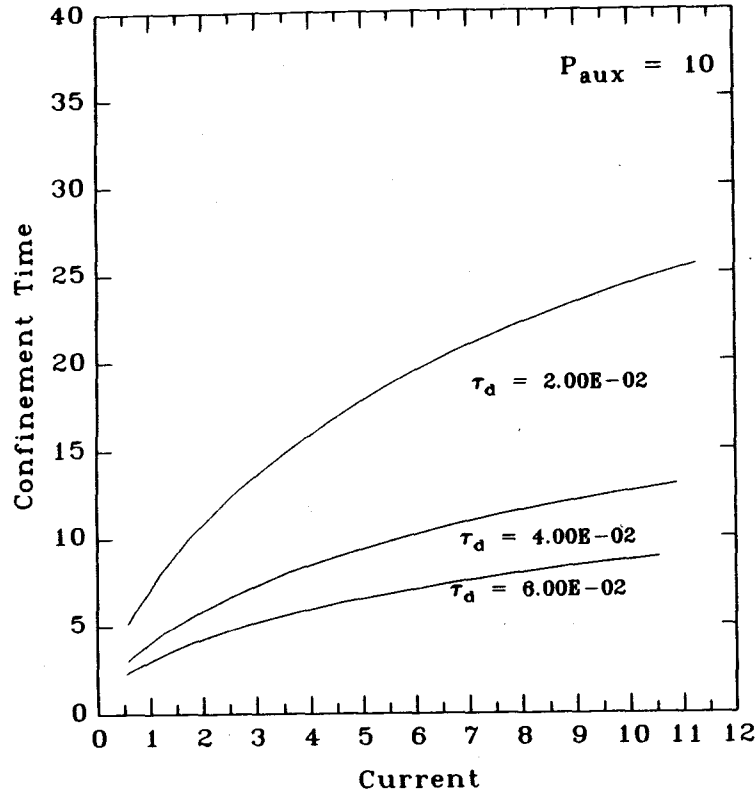


FIGURE 5 Normalized energy confinement time versus plasma current  $I_p$  (in MA), for fixed  $P_{aux} = 10$  MW. Equation (13b) was used, setting  $P = P_{aux}/kI_p^{0.8}$ . (CIT parameters as given in Appendix A are assumed to determine  $k$  in  $P_0 = kI_p^{0.8}$ ).

#### APPENDIX C ON CALCULATION OF STORED ENERGY INCLUDING PARTICLE SOURCES FOR MULTIPLE SPECIES

Defining

$$\bar{W} = L \int_0^a dr r \sum n_j T_j, \quad L = 4\pi^2 R_0 \quad (C.1)$$

$$\frac{\bar{W}}{L} = \int dr \frac{r^2}{2} \sum [n_j (-T_j') + T_j (-n_j')].$$

The energy and particle balances are written as

$$m_j \chi_j (-T_j') + r q_{cj} = \int_0^r dr' r' s_{Ej}(r') \equiv S_{Ej}(r) \quad (C.2a)$$

$$r D_j (-n_j') + r \Gamma_{cj} = \int_0^r dr' r' s_{pj}(r') = S_{pj}(r) \quad (C.2b)$$

where  $\Gamma_{cj}$  is the purely convective particle flux piece, vs.  $-D_j n_j'$ , the diffusive piece.

We define

$$y_{Ej}(r) = \int_0^r \frac{dr' r'}{2\chi_j(r')}, \quad y_{Ej}(a) \equiv \bar{\tau}_{Ej} \sim \frac{a^2}{4\chi_j} \quad (C.3a)$$

$$y_{pj}(r) = \int_0^r \frac{dr' r'}{2D_j(r')}, \quad y_{pj}(a) \equiv \bar{\tau}_{pj} \sim \frac{a^2}{4D_j}, \quad (C.3b)$$

and obtain from Eqs. (C.1), (C.2), (C.3)

$$\begin{aligned} \bar{W} = L \sum_j \left\{ \bar{\tau}_{Ej} \int_0^a dr r s_{Ej} \left( 1 - \frac{y_{Ej}(r)}{y_{Ej}(a)} \right) \right. \\ \left. + \bar{\tau}_{pj} \int_0^a dr r T_j s_{jp} \left( 1 - \frac{y_{pj}(r)}{y_{pj}(a)} \right) \right. \\ \left. - \int_0^a dr r \left( \frac{dy_{Ej}}{dr} q_{cj} + \frac{dy_{pj}}{dr} \Gamma_{cj} T_j \right) \right\}. \end{aligned} \quad (C.4)$$

The energy confinement time is

$$\tau_E = \bar{W} / \int_0^a dr r s_{Ej}(r), \quad (C.5)$$

a straightforward extension of Eqs. (20a,b) using a slightly changed notation for the energy source terms. We note that the particle source term  $T_j s_{pj}$  helps to increase the stored energy, in addition

to  $s_{Ej}$ . Thus, pellet fuelling may appear to increase  $\tau_E$  through the second term in (C.4).

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## Anomalous Transport in the Second Stability Regime

Operation in the second stability regime can manifest itself as an improvement of confinement with  $\beta$ . This will be the case if the main cause of confinement deterioration with increasing  $\beta$  is pressure-driven instabilities. Stellarators are well suited to the experimental study of these effects because of their ability to operate with zero toroidal current, eliminating one source of free energy. On the basis of resistive pressure-gradient-driven turbulence theory, an improved confinement regime that is accessible at relatively low values of  $\beta$  is predicted for the Advanced Toroidal Facility (ATF).

**Key Words:** *second stability regime, anomalous transport, fluctuation level, pressure-gradient-driven turbulence, stellarator confinement*

### 1. INTRODUCTION

Toroidal confinement devices show deterioration of confinement when heated with high levels of auxiliary heating power. Experimental observations have not yet revealed whether the cause of the confinement deterioration is directly related to the injected power or is correlated with  $\beta$ . However, as this effect is present with all sources of heating power, it is logical to assume the latter. Under this assumption, pressure-driven modes are good candidates to explain the anomalous particle and thermal diffusivities measured in high- $\beta$  plasmas.

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